

A PROPERTY OF ERGODIC FLOWS

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ABSTRACT. In this paper we introduce a property of ergodic flows, called Property B. We prove that any ergodic hyperfinite equivalence relation of type III_0 whose associated flow satisfies this property is not of product type. A consequence of this result is that any properly ergodic flow with Property B is not approximately transitive. We use Property B to construct a non-AT flow which - up to conjugacy - is a flow built under a function with the dyadic odometer as base automorphism.

1. INTRODUCTION

A remarkable result of Krieger [9] establishes a complete correspondence between orbit equivalence classes of ergodic hyperfinite equivalence relations of type III_0 , conjugacy classes of properly ergodic flows and isomorphism classes of approximately finite dimensional factors of type III_0 . Product type equivalence relations are hyperfinite equivalence relations, which, up to orbit equivalence, are generated by product type odometers. In order to show that there exist ergodic non-singular automorphisms not orbit equivalent to any product type odometer, Krieger [7] introduced a property of non-singular automorphisms, called Property A. He proved that any product type odometer satisfies this property [8], and he also constructed an ergodic non-singular automorphism that does not have this property, and therefore is not of product type. It was shown in [11] that there exist non singular automorphisms which satisfy Property A but which are not of product type.

To characterize the ITPFI factors among all approximately finite dimensional factors, Connes and Woods [1] introduced a property of ergodic actions, called approximate transitivity, shortly AT. They showed that an approximately finite dimensional factor of type III_0 is an ITPFI factor if and only if its flow of weights is AT. Equivalently, their result

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says that an ergodic hyperfinite equivalence relation \mathcal{R} of type III_0 is of product type if and only if the associated flow of \mathcal{R} is AT.

In this paper we introduce a property of ergodic flows, called Property B. We show that any properly ergodic flow with this property is not AT and we construct a flow which has this property. The non AT flow corresponding to the non ITPFI factor constructed in [4] does not have Property B, and so the property of a flow to be not AT is not equivalent to Property B.

The paper is organized as follows. In Section 2, we recall some notations and definitions. In Section 3 we define Property B, we show that this property is invariant for conjugacy of flows and we characterize this property for a flow built under a function. In Section 4 we prove that a hyperfinite ergodic equivalence relation \mathcal{R} of type III_0 whose associated flow satisfies Property B is not of product type and we show that a properly ergodic flow which has Property B is not AT. In Section 5, we show that there exists a flow with Property B. This flow is built under a function with the dyadic odometer as base automorphism.

2. PRELIMINARIES

Throughout this paper (X, \mathfrak{B}, μ) will be a standard σ -finite measure space. A measurable flow on (X, \mathfrak{B}, μ) is a one parameter group of nonsingular automorphisms $\{F_t\}_{t \in \mathbb{R}}$ of (X, \mathfrak{B}, μ) such that the mapping $X \times \mathbb{R} \ni (x, t) \mapsto F_t(x) \in X$ is measurable. Two flows $\{F_t\}_{t \in \mathbb{R}}$ and $\{F'_t\}_{t \in \mathbb{R}}$ on (X, \mathfrak{B}, μ) and $(X', \mathfrak{B}', \mu')$ respectively, are conjugate if there exists an isomorphism $T : (X, \mathfrak{B}, \mu) \rightarrow (X', \mathfrak{B}', \mu')$ such that for all $t \in \mathbb{R}$ and for μ -almost all $x \in X$, $F'_t(T(x)) = T(F_t(x))$. We say that $\{F_t\}_{t \in \mathbb{R}}$ is ergodic if any F_t -invariant measurable set is either null or conull.

Let \mathcal{R} be an equivalence relation on (X, \mathfrak{B}, μ) . We say that \mathcal{R} is a countable measured equivalence relation if the equivalence classes $\mathcal{R}(x)$, $x \in X$ are countable, \mathcal{R} is a measurable subset of $X \times X$, and the saturation of any set of measure zero has measure zero. \mathcal{R} is called ergodic if any invariant set is either null or conull. Recall that if ν_l and ν_r are the left and the right counting measures on \mathcal{R} we have that $\nu_l \sim \nu_r$ and $\delta(x, y) = \frac{d\nu_l}{d\nu_r}(x, y)$ is the Radon-Nikodym cocycle of μ with respect to \mathcal{R} . We say that the measure μ is lacunary if there exist $\varepsilon > 0$ such that $\delta(x, y) = 0$ or $|\delta(x, y)| > \varepsilon$, for $(x, y) \in \mathcal{R}$. The full group $[\mathcal{R}]$ of \mathcal{R} is the group of all nonsingular automorphisms V of (X, \mathfrak{B}, μ) with $(x, Vx) \in \mathcal{R}$ for μ -a.e. $x \in X$.

A countable measured equivalence relation \mathcal{R} is called finite if $\mathcal{R}(x)$ are finite for almost all $x \in X$. We say that \mathcal{R} is hyperfinite, if there

are finite relations \mathcal{R}_n with $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$ and $\cup \mathcal{R}_n = \mathcal{R}$, up to a set of measure zero. We recall that \mathcal{R} is hyperfinite if and only if there exists a nonsingular automorphism T on (X, \mathcal{B}, μ) such that, up to a set of measure zero, \mathcal{R} is equal to the equivalence relation $\mathcal{R}_T = \{(x, T^n x), x \in X, n \in \mathbb{Z}\}$ generated by T , that is, $\mathcal{R}(x) = \{T^n x, n \in \mathbb{Z}\}$, for μ -a.e. $x \in X$.

Two countable measured equivalence relations \mathcal{R} and \mathcal{R}' on (X, \mathfrak{B}, μ) and $(X', \mathfrak{B}', \mu')$ respectively, are called orbit equivalent if there exists an isomorphism $S : (X, \mathfrak{B}, \mu) \rightarrow (X', \mathfrak{B}', \mu')$, such that $S(\mathcal{R}(x)) = \mathcal{R}'(Sx)$ for μ -a.e. $x \in X$.

Let $(k_n)_{n \geq 1}$ be a sequence of positive integers, with $k_n \geq 2$. Consider the infinite product probability space $(X, \mu) = \prod_{n=1}^{\infty} (X_n, \mu_n)$, where $X_n = \{0, 1, \dots, k_n - 1\}$ and μ_n are probability measures on X_n such that $\mu_n(x) > 0$, for all $x \in X_n$. We recall that the tail equivalence relation \mathcal{T} on (X, μ) is defined for $x = (x_n)_{n \geq 1}$ and $y = (y_n)_{n \geq 1}$ by

$$(x, y) \in \mathcal{T} \text{ iff there exists } n \geq 1 \text{ such that } x_i = y_i \text{ for all } i > n.$$

It easily can be observed that, up to a set of measure zero, \mathcal{T} is generated by the odometer defined on (X, μ) . A countable measured equivalence relation is said of product type if it is orbit equivalent to the tail equivalence relation on an infinite product probability space as above, or equivalently, if it is orbit equivalent to the equivalence relation generated by a product type odometer.

An ergodic equivalence relation \mathcal{R} is of type III if there is no σ -finite \mathcal{R} -invariant measure ν equivalent to μ . The type III equivalence relations are further classified in subtypes III_λ , where $0 \leq \lambda \leq 1$. Up to orbit equivalence, for $\lambda \neq 0$, there is only one hyperfinite equivalence of type III_λ , and this is of product type.

The orbit equivalence classes of ergodic hyperfinite equivalence relations of type III_0 are completely classified by the conjugacy class of their associated flow. For more details we refer the reader to [3] and [12].

In order to show that there exists ergodic non-singular automorphisms not orbit equivalent to any product odometer, Krieger introduced a property of non-singular automorphisms, called Property A. This property can be defined for equivalence relations (see [10]), as follows. Suppose that \mathcal{R} is a hyperfinite equivalence relation on (X, \mathfrak{B}, μ) . Let ν be a σ -finite measure on X , equivalent to μ , and δ_ν the corresponding Radon-Nicodym cocycle. For $x \in A$, define

$$\Lambda_{\nu, A, \mathcal{R}}(x) = \{\log \delta_\nu(y, x) : (x, y) \in \mathcal{R} \text{ and } y \in A\}$$

$$= \left\{ \log \frac{d\nu \circ \phi}{d\nu}(x) : \phi \in [\mathcal{R}], (x, \phi(x)) \in \mathcal{R} \text{ and } \phi(x) \in A \right\}.$$

For a σ -finite measure $\nu \sim \mu$, $A \in \mathcal{B}$ of positive measure and $s, \zeta > 0$, set

$$K_{\nu, \mathcal{R}}(A, s, \zeta) = \{x \in A : (e^{s-\zeta}, e^{s+\zeta}) \cap \Lambda_{\nu, A, \mathcal{R}}(x) \neq \emptyset\} \cup \\ \{x \in A : (-e^{s+\zeta}, -e^{s-\zeta}) \cap \Lambda_{\nu, A, \mathcal{R}}(x) \neq \emptyset\}.$$

Definition 1. Let \mathcal{R} be a hyperfinite equivalence relation on (X, \mathfrak{B}, μ) . Then \mathcal{R} has Property A if there exists a σ -measure $\nu \sim \mu$ and $\eta, \zeta > 0$ such that: every set $A \in \mathfrak{B}$ of positive measure contains a set $B \in \mathfrak{B}$ of positive measure such that

$$\limsup_{s \rightarrow \infty} K_{\nu, \mathcal{R}}(B, s, \zeta) > \eta \cdot \nu(B).$$

If \mathcal{R} is a hyperfinite equivalence relation and T is a non-singular automorphism such that $\mathcal{R} = \mathcal{R}_T$, up to a null set, it easily can be observed that \mathcal{R} has Property A if and only if T has Property A (see [11]). We mention the following result (see [8] and [11]) that will be used in this paper.

Proposition 2.1. Assume that \mathcal{R} has Property A. Then there exist $\eta, \delta > 0$ such that for all $\lambda \sim \mu$ and all $\epsilon > 0$, every measurable set A of positive measure contains a measurable set B of positive measure with

$$\limsup_{s \rightarrow \infty} K_{\lambda, \mathcal{R}}(B, s, \delta + \epsilon) > e^{-\epsilon} \eta \cdot \lambda(B).$$

We recall that Krieger's result from [8], can be reformulated in the following way [11]:

Theorem 2.2. Any ergodic equivalence relation of product type and of type III has Property A.

3. PROPERTY B

In this section we define a property of measurable flows that we call Property B, we show that this is an invariant for conjugacy of flows, and we characterize this property for a flow built under a function.

Let $\{F_t\}_{t \in \mathbb{R}}$ be a flow of automorphisms of (X, \mathfrak{B}, μ) . For $A \in \mathfrak{B}$ of positive measure and $s, \delta > 0$ we define

$$\Lambda_{F, \delta, s}(A) = \{x \in A, \exists t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}), F_t(x) \in A\}.$$

Definition 2. We say that $\{F_t\}_{t \in \mathbb{R}}$ has Property B if there exists a measurable set $A \subseteq X$ of positive measure such that for all $\delta > 0$

$$(1) \quad \limsup_{s \rightarrow \infty} \mu(\Lambda_{F, \delta, s}(A)) = 0.$$

Proposition 3.1. *Let $\{F_t\}_{t \in \mathbb{R}}$ be a flow on (X, \mathfrak{B}, μ) satisfying Property B, and μ' a σ -finite measure equivalent to μ . Then, the flow $\{F_t\}_{t \in \mathbb{R}}$ on (X, \mathfrak{B}, μ') has Property B.*

Proof. Let $\mu' \sim \mu$ be a σ -finite measure equivalent to μ and denote by f the Radon-Nikodym derivative of μ' with respect to μ . Thus, $\mu'(A) = \int_A f d\mu$ whenever $A \in \mathfrak{B}$.

Let A be a measurable set satisfying (1). There exists a positive integer k such that $\mu(A \cap \{x \in X, f(x) < k\}) > 0$. Hence, $B = A \cap \{x \in X, f(x) < k\}$ is a subset of A of positive measure, and then, for every $\delta > 0$

$$\limsup_{s \rightarrow \infty} \mu(\Lambda_{F, \delta, s}(B)) = 0.$$

Since $\mu(B) > 0$ and $\mu \sim \mu'$ it results that $\mu'(B) > 0$. Notice that

$$\mu'(\Lambda_{F, \delta, s}(B)) = \int_{\Lambda_{F, \delta, s}(B)} f d\mu < k \int_{\Lambda_{F, \delta, s}(B)} d\mu = k \cdot \mu(\Lambda_{F, \delta, s}(B)).$$

Consequently,

$$\limsup_{s \rightarrow \infty} \mu'(\Lambda_{F, \delta, s}(B)) = 0.$$

and therefore, the flow $\{F_t\}_{t \in \mathbb{R}}$ on (X, \mathfrak{B}, μ') has Property B. \square

Proposition 3.2. *Let $T : (X', \mathfrak{B}', \mu') \rightarrow (X, \mathfrak{B}, \mu)$ be an isomorphism and assume that $\mu' = \mu \circ T^{-1}$. If $\{F_t\}_{t \in \mathbb{R}}$ is a flow on (X, \mathfrak{B}, μ) that satisfies Property B and $\{F'_t\}_{t \in \mathbb{R}}$ is a flow on $(X', \mathfrak{B}', \mu')$ such that $T(F'_t(x)) = F_t(Tx)$, for all $t \in \mathbb{R}$ and for μ' -almost all $x \in X'$, then F'_t has Property B.*

Proof. Assume that there exists a measurable subset A of X of positive measure which satisfies (1). Let $\delta, s > 0$. Up to sets of measure zero, the following equalities hold:

$$\begin{aligned} & \Lambda_{F', \delta, s}(T^{-1}(A)) \\ &= \{x \in T^{-1}(A), \exists t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}), F'_t(x) \in T^{-1}(A)\} \\ &= \{x \in X', Tx \in A, \exists t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}), T(F'_t(x)) \in A\} \\ &= \{x \in X', Tx \in A, \exists t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}), F_t(Tx) \in A\} \\ &= T^{-1}(\{y \in A, \exists t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}), F_t(y) \in A\}) \\ &= T^{-1}(\Lambda_{F, \delta, s}(A)). \end{aligned}$$

Hence,

$$\mu'(\Lambda_{F', \delta, s}(T^{-1}(A))) = \mu' \circ T^{-1}(\Lambda_{F, \delta, s}(A)) = \mu(\Lambda_{F, \delta, s}(A)).$$

It then follows that for every $\delta > 0$, we have

$$\limsup_{s \rightarrow \infty} \mu'(\Lambda_{F', \delta, s}(T^{-1}(A))) = \limsup_{s \rightarrow \infty} \mu(\Lambda_{F, \delta, s}(A)) = 0,$$

and therefore, the flow $\{F'_t\}_{t \in \mathbb{R}}$ on $(X', \mathfrak{B}', \mu')$ has Property B. \square

We can prove now the following result:

Proposition 3.3. *Property B is an invariant for conjugacy of flows.*

Proof. Let (X, \mathfrak{B}, μ) , $(X', \mathfrak{B}', \mu')$ be two σ -finite measure spaces and assume that $\{F_t\}_{t \in \mathbb{R}}$ is a flow on (X, \mathfrak{B}, μ) which satisfies Property B. Let $\{F'_t\}_{t \in \mathbb{R}}$ be a flow on $(X', \mathfrak{B}', \mu')$ which is conjugate to $\{F_t\}_{t \in \mathbb{R}}$. Hence, there exists an isomorphism $T : (X', \mathfrak{B}', \mu') \rightarrow (X, \mathfrak{B}, \mu)$ such that $F_t(Tx) = T(F'_t(x))$ for μ' -almost all $x \in X'$ and for all $t \in \mathbb{R}$. As T is an isomorphism, $\mu' \sim \mu \circ T^{-1}$. Let μ'' be the measure on X' given by $\mu'' = \mu \circ T^{-1}$. Thus $F_t(Tx) = T(F'_t(x))$ for μ'' almost all $x \in X'$ and for all $t \in \mathbb{R}$. By Proposition 3.2, we have that $\{F'_t\}_{t \in \mathbb{R}}$ on $(X', \mathfrak{B}', \mu'')$ has Property B. As μ'' and μ' are equivalent measures, Proposition 3.1 implies that $\{F'_t\}_{t \in \mathbb{R}}$ has Property B. \square

Let T be an automorphism of $(X_0, \mathfrak{B}_0, \mu_0)$ and $\xi : X_0 \rightarrow \mathbb{R}$ be a positive measurable function. Consider $Y = \{(x, t) \in X_0 \times \mathbb{R}, 0 \leq t < \xi(x)\}$ and let ν be the measure on Y that is the restriction of the product measure $\mu_0 \times \lambda$, where λ is the usual Lebesgue measure on \mathbb{R} . Let $\{F_t\}_{t \in \mathbb{R}}$ be the flow built under the function ξ with base automorphism T ; it is defined on (Y, ν) , and for $t > 0$ is given by

$$F_t(x, s) = \begin{cases} (x, t + s) & \text{if } 0 \leq t + s < \xi(x) \\ (T(x), t + s - \xi(x)) & \text{if } \xi(x) \leq t + s < \xi(T(x)) + \xi(x) \\ \dots & \end{cases}$$

For a measurable set $A \subseteq X_0$ we define

$$\Delta_{F, \delta, s}(A) = \{x \in A, \exists t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}), F_t(x, 0) \in A \times \{0\}\}.$$

With this notation we have the following result:

Proposition 3.4. *The flow $\{F_t\}_{t \in \mathbb{R}}$ has Property B, if and only if there exists a measurable set $A_0 \subseteq X_0$ of positive measure such that, for all $\delta > 0$,*

$$(2) \quad \limsup_{s \rightarrow \infty} \mu_0(\Delta_{F, \delta, s}(A_0)) = 0.$$

Proof. Assume that $\{F_t\}_{t \in \mathbb{R}}$ has Property B. Then, there exists a measurable set $A \subseteq Y$ such that, for every $\delta > 0$,

$$\limsup_{s \rightarrow \infty} \nu(\Lambda_{F, 2\delta, s}(A)) = 0.$$

Since A has positive measure, there exists a measurable set $A_0 \subseteq X_0$ of positive measure, an integer $m \geq 1$, and a positive real α such that for all $x \in A_0$, $\lambda(A_x \cap [m, m+1]) > \alpha$, where $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$. Let $K = A_0 \times [m, m+1] \cap A$. Clearly, $K \subseteq A$, and then, for all $\delta > 0$, we have

$$(3) \quad \limsup_{s \rightarrow \infty} \nu(\Lambda_{F, 2\delta, s}(K)) = 0.$$

Let $\delta > 0$. For any $x \in \Delta_{F, \delta, s}(A_0)$, there exists $y \in A_0$ and $t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta})$ such that $F_t(x, 0) = (y, 0)$. For $(x, a), (y, b) \in K$, we have that

$$F_{t-a+b}(x, a) = F_{t+b}(x, 0) = F_b(F_t(x, 0)) = F_b(y, 0) = (y, b).$$

It is straightforward to check that for s large enough, $t - a + b \in (e^{s-2\delta}, e^{s+2\delta}) \cup (-e^{s+2\delta}, -e^{s-2\delta})$ whenever $t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta})$. Consequently,

$$\Delta_{F, \delta, s}(A_0) \times [m, m+1] \cap K \subseteq \Lambda_{F, 2\delta, s}(K),$$

whence

$$\alpha \cdot \mu_0(\Delta_{F, \delta, s}(A_0)) \leq \nu(\Lambda_{F, 2\delta, s}(K)),$$

and then (2) follows from (3).

Conversely, consider $A_0 \subseteq X_0$ satisfying (2). Let $A = A_0 \times [0, 1] \cap Y$. Proceeding in the same manner as above, for all $\delta > 0$, and for s large enough, we have

$$\Lambda_{F, \delta, s}(A) \subseteq \Delta_{F, 2\delta, s}(A_0) \times [0, 1] \cap A.$$

Then, by (2), we obtain that $\{F_t\}_{t \in \mathbb{R}}$ satisfies Property B. □

4. PROPERTY B IMPLIES NOT AT

In this section we show that if \mathcal{R} is an ergodic hyperfinite equivalence relation of type III_0 whose associated flow has Property B, then \mathcal{R} does not satisfy Krieger's Property A and therefore is not of product type. A consequence of this result is that any properly ergodic flow with Property B is not approximately transitive. Remark that if \mathcal{R} is of type III_λ , $\lambda \neq 0$, then the associated flow of \mathcal{R} does not have Property B.

Consider an ergodic hyperfinite equivalence relation \mathcal{R} of type III_0 on (X, \mathfrak{B}, μ) and let δ be the Radon-Nicodym cocycle of μ with respect to \mathcal{R} . Replacing eventually μ with an equivalent measure we can assume that μ is a lacunary measure (see for example [6], Proposition 2.3). Define

$$\xi(x) = \min\{\log \delta(x', x); (x', x) \in \mathcal{R}, \log \delta(x', x) > 0\}$$

and consider \mathcal{S} the equivalence relation on X given by

$$(x, y) \in \mathcal{S} \text{ if and only if } (x, y) \in \mathcal{R} \text{ and } \delta(x, y) = 1.$$

Let $\mathfrak{B}(\mathcal{S})$ the σ -algebra of sets in \mathfrak{B} that are \mathcal{S} -invariant. Let X_0 be the quotient space $X/\mathfrak{B}(\mathcal{S})$, that is the space of ergodic components of \mathcal{S} . We denote the quotient map from X onto X_0 by π , where $\pi(x)$ is the element of X_0 containing x . On X_0 , consider the measure $\mu_0 = \mu \circ \pi^{-1}$. Note that $\xi(x)$ is $\mathfrak{B}(\mathcal{S})$ -measurable and therefore, ξ can be regarded as a function on X_0 . We have an ergodic automorphism T on X_0 defined $T(\pi(x)) = \pi(x')$ where $(x, x') \in \mathcal{R}$ and $\log \delta(x', x) = \xi(\pi(x))$. Then, the associated flow $\{F_t\}_{t \in \mathbb{R}}$ of \mathcal{R} can be realized as the flow built under the ceiling function ξ with base automorphism T (see for example [5] or [6]).

Lemma 4.1. *Let $(x, x') \in \mathcal{R}$, $z = \pi(x)$ and $z' = \pi(x')$. Then $F_{\log \delta(x', x)}(z, 0) = (z', 0)$.*

Proof. Notice that it is enough to prove the lemma for $\log \delta(x', x)$ positive. Since μ is a lacunary measure, there are only finitely many values, say n , of $\log \delta(z, x)$ between 0 and $\log \delta(x', x)$. Hence there exists x_1, x_2, \dots, x_n in the orbit $\mathcal{R}(x)$ of x such that $0 < \log \delta(x_1, x) < \dots < \log \delta(x_n, x) < \log \delta(x', x)$. Then

$$\log \delta(x', x) = \log \delta(x_1, x) + \log \delta(x_2, x_1) + \dots + \log \delta(x_n, x_{n-1}) + \log \delta(x', x_n).$$

If $z_i = \pi(x_i)$, then $z_i = T^i(z)$ and $F_{\xi(T^{i-1}(z))}(T^{i-1}(z), 0) = (T^i(z), 0)$, for $1 \leq i \leq n$. Notice that $\log \delta(x', x) = \xi(z) + \xi(T(z)) + \dots + \xi(T^n(z))$. Therefore

$$\begin{aligned} F_{\log \delta(x', x)}(z, 0) &= F_{\xi(z) + \xi(T(z)) + \dots + \xi(T^n(z))}(z, 0) \\ &= F_{\xi(T(z)) + \dots + \xi(T^n(z))}(T(z), 0) = \dots = (z', 0). \end{aligned}$$

□

Theorem 4.2. *With the above notation, if the associated flow $\{F_t\}_{t \in \mathbb{R}}$ of \mathcal{R} has Property B, then \mathcal{R} does not have Property A.*

Proof. By Proposition 3.4, we can find a measurable set $A_0 \subseteq X_0$ of positive measure such that, for all $\delta > 0$,

$$\limsup_{s \rightarrow \infty} \mu_0(\Delta_{F, \delta, s}(A_0)) = 0.$$

Let $C = \pi^{-1}(A_0) \subseteq X$ and $\delta > 0$. Consider $s > 0$ and $x \in K_{\mu, \mathcal{R}}(C, s, \delta)$. Thus, $x \in C$ and there exists $y \in C$ such that $(x, y) \in \mathcal{R}$ and $\log \delta(y, x) \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta})$. From Lemma 4.1, we have that

$$F_{\log \delta(y, x)}(\pi(x), 0) = (\pi(y), 0).$$

Hence, $\pi(x) \in \Delta_{F,\delta,s}(A_0)$ and then, $x \in \pi^{-1}(\Delta_{F,\delta,s}(A_0))$. Therefore,

$$K_{\mu,\mathcal{R}}(C, s, \delta) \subseteq \pi^{-1}(\Delta_{F,\delta,s}(A_0)),$$

and consequently,

$$\mu(K_{\mu,\mathcal{R}}(C, s, \delta)) \leq \mu \circ \pi^{-1}(\Delta_{F,\delta,s}(A_0)) = \mu_0(\Delta_{F,\delta,s}(A_0)).$$

This clearly implies that

$$\limsup_{s \rightarrow \infty} \mu(K_{\mu,\mathcal{R}}(C, s, \delta)) = 0$$

and then, by Proposition 2.1, \mathcal{R} does not have Property A. \square

Remark 1. *Since any product type equivalent relation of type III satisfies Property A, it follows that an equivalent relation \mathcal{R} whose associated flow has Property B is not of product type.*

Recall that any properly ergodic flow is the associated flow of certain ergodic hyperfinite equivalence relation of type III_0 and a hyperfinite ergodic equivalence relation is of product type, if and only if the associated flow is approximately transitive. We have then the following result:

Corollary 4.3. *Let $\{F_t\}_{t \in \mathbb{R}}$ be a properly ergodic flow on (X, \mathfrak{B}, μ) which satisfies Property B. Then $\{F_t\}_{t \in \mathbb{R}}$ is not approximately transitive.*

Remark 2. *There exists ergodic flows which are not AT and do not satisfy Property B, as the following example shows.*

Example 1. *In [4], Giordano and Handelman constructed a factor N whose flow of weights is not AT. We recall that the flow of weights of N can be realized as the flow built under a constant function and which has a base automorphism that can be identified with the Poisson boundary of the matrix valued random walk corresponding to the dimension space given by the sequence of matrices*

$$\begin{bmatrix} x^{5^n} & 1 \\ 1 & x^{5^n} \end{bmatrix}, n \geq 1.$$

Since, up to isomorphism, N is the von Neumann algebra associated to an ergodic hyperfinite equivalence relation \mathcal{R} , the flow of weights of N is, up to conjugacy, the associated flow of \mathcal{R} . According to [11], the equivalence relation \mathcal{R} has Property A, and then, from Theorem 4.2 we conclude that the associated flow of \mathcal{R} does not satisfy Property B.

The following result gives a sufficient condition for a nonsingular automorphism to be not AT.

Corollary 4.4. *Let T be a nonsingular automorphism of (X, \mathfrak{B}, μ) . Assume that there exists $A \subset X$ of positive measure such that*

$$\limsup_{s \rightarrow \infty} \{x \in A : \exists n \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}), T^n x \in A\}.$$

Then T is not AT.

Proof. Proposition 3.4 implies that $\{F_t\}_{t \in \mathbb{R}}$, the flow built under the constant function $f = 1$ with base automorphism T has Property B and then by Corollary 4.3 it follows that $\{F_t\}_{t \in \mathbb{R}}$ is not AT. From Lemma 2.5 of [1], we conclude that T is not AT. \square

5. AN ERGODIC FLOW WHICH SATISFIES PROPERTY B

In this section we construct a properly ergodic flow which satisfies Property B and therefore is not AT. The flow that we construct is a flow built under a function with a product odometer (conjugate to the dyadic odometer) as base automorphism.

Let $(z_n)_{n \geq 1}$ be the sequence of integers given by $z_n = 2^n - 1$, for $n \geq 1$. Consider the product space $X = \prod_{n \geq 1} \{0, 1, \dots, z_n\}$ endowed with the usual product σ -algebra and the product measure $\mu = \otimes_{n \geq 1} \mu_n$, where μ_n are the probability measures on $\{0, 1, \dots, z_n\}$ given by $\mu_n(i) = \frac{1}{2^n}$, for $i = 0, 1, \dots, z_n$ and $n \geq 1$. Let $T : X \rightarrow X$ be the product odometer defined on X . We recall that T is the nonsingular automorphism defined for almost every $x \in X$ by

$$(4) \quad (Tx)_n = \begin{cases} 0 & \text{if } n < N(x), \\ x_n + 1 & \text{if } n = N(x), \\ x_n & \text{if } n > N(x), \end{cases}$$

where $N(x) = \min\{n \geq 1 : x_n < z_n\}$. Notice that T is measure conjugate to the dyadic odometer.

Let $(K_n)_{n \geq 4}$ be the sequence given by

$$K_n = 1!2! \cdots n!, \text{ for } n \geq 4,$$

and let $f : X \rightarrow \mathbb{R}$ be the function defined for almost every $x \in X$ by setting

$$(5) \quad f(x) = K_{2^{N+1} + x_{N+1}}$$

where $x = (x_n)_{n \geq 1}$ and $N = N(x)$.

Proposition 5.1. *Let $n \geq 4$ be a positive integer, $m = \lfloor \log_2 n \rfloor$ and $l = n - 2^m$. For almost every $x \in X$, we have:*

(i) If there exists an integer $k \geq 1$ such that

$$(6) \quad K_n \leq \sum_{i=0}^{k-1} f(T^i x) < K_{n+1},$$

then $x_{m+1} = l$.

(ii) If there exists an integer $k \geq 1$ such that

$$(7) \quad K_n \leq \sum_{i=1}^k f(T^{-i} x) < K_{n+1},$$

then $x_{m+1} = l$.

Proof. (i) Let $x \in X$ such that $K_n \leq \sum_{i=0}^{k-1} f(T^i x) < K_{n+1}$, for some integer $k \geq 1$. Let

$$p = \max\{N(T^i x); 0 \leq i \leq k-1\}.$$

Hence, there exists j , $0 \leq j < k$ such that $N(T^j x) = p$. By (5) we have that $f(T^j x) = K_{2^{p+1}+x_{p+1}}$. From (4) we deduce that $(T^i x)_n = x_n$ for $n > p$ and $1 < i \leq k$. Also, (4) implies that $2 \cdot 2^2 \cdots 2^p > k$. Hence,

$$K_{2^{p+1}+x_{p+1}} \leq \sum_{i=0}^{k-1} f(T^i x) < 2 \cdot 2^2 \cdots 2^p \cdot K_{2^{p+1}+x_{p+1}} < K_{2^{p+1}+x_{p+1}+1}.$$

We claim that $n = 2^{p+1} + x_{p+1}$. Indeed, if $n < 2^{p+1} + x_{p+1}$ we have $K_{n+1} \leq K_{2^{p+1}+x_{p+1}} \leq \sum_{i=0}^{k-1} f(T^i x)$, which contradicts (6). If $n > 2^{p+1} + x_{p+1}$, then $K_n \geq K_{2^{p+1}+x_{p+1}+1} > \sum_{i=0}^{k-1} f(T^i x)$, which again contradicts (6). Therefore $n = 2^{p+1} + x_{p+1}$, and then $m = p$ and $x_{m+1} = l$.

(ii) Let $x \in X$ such that $K_n \leq \sum_{i=1}^k f(T^{-i} x) < K_{n+1}$, for some positive integer k . Let

$$p = \max\{N(T^{-i} x); 1 \leq i \leq k\}$$

and remark that $(T^{-i} x)_n = x_n$ for $n > p$ and $1 \leq i \leq k$. The proof follows in the same way as in case (i) and we leave the details to the reader. \square

Let $\{F_t\}_{t \in \mathbb{R}}$ be the flow built under the function f with base automorphism T . Notice that $\{F_t\}_{t \in \mathbb{R}}$ is a properly ergodic flow. The following lemma follows directly from the definition of $\{F_t\}_{t \in \mathbb{R}}$.

Lemma 5.2. (i) If $t > 0$ then $F_t(x, 0) \in X \times \{0\}$ if and only if there exists an integer $k \geq 1$ such that $t = \sum_{i=0}^{k-1} f(T^i x)$.

(ii) If $t < 0$ then $F_t(x, 0) \in X \times \{0\}$ if and only if there exists an integer $k \geq 1$ such that $t = -\sum_{i=1}^k f(T^{-i} x)$.

Proposition 5.3. *For any $\delta > 0$,*

$$(8) \quad \lim_{s \rightarrow \infty} \mu(\Delta_{F,\delta,s}(X)) = 0.$$

Proof. By Lemma 5.2 we have

$$(9) \quad \mu(\Delta_{F,\delta,s}) = \mu \left(\left\{ x \in X; \exists k \in \mathbb{N}, e^{s-\delta} < \sum_{i=0}^{k-1} f(T^i x) < e^{s+\delta} \right\} \cup \right. \\ \left. \left\{ x \in X; \exists k \in \mathbb{N}, e^{s-\delta} < \sum_{i=1}^k f(T^{-i} x) < e^{s+\delta} \right\} \right).$$

Proposition 5.1 implies that

$$(10) \quad \mu \left(\left\{ x \in X; \exists k \in \mathbb{N}, K_n \leq \sum_{i=0}^{k-1} f(T^i x) < K_{n+1} \right\} \right) \leq \frac{1}{2^{\lfloor \log_2 n \rfloor + 1}},$$

$$(11) \quad \mu \left(\left\{ x \in X; \exists k \in \mathbb{N}, K_n \leq \sum_{i=1}^k f(T^{-i} x) < K_{n+1} \right\} \right) \leq \frac{1}{2^{\lfloor \log_2 n \rfloor + 1}}.$$

Notice that for s sufficiently large, $(e^{s-\delta}, e^{s+\delta})$ intersects at most two consecutive intervals $[K_n, K_{n+1})$. This, together (9), (10) and (11) implies (8). \square

From Proposition 3.4 and Proposition 5.3 we can then conclude:

Corollary 5.4. *The flow $\{F_t\}_{t \in \mathbb{R}}$ constructed above satisfies Property B.*

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